Optimal and Markov-perfect Nash equilibria in harvesting age-structured populations

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Abstract

We specify an analytically solvable age-structured harvesting model for collectively optimal and Markov-perfect Nash equilibria in both deterministic and stochastic settings. The model has any number of age-classes and is assumed to be harvested from one or two age classes. The collectively optimal harvests are obtained in closed form as functions of the number of individuals in the given age class. The existence of sustainable solutions is shown to depend on fundamental biological factors and rate of discount in addition to the internal delays in the age-structured system. In a symmetric game all actors harvest both age classes and the existence of sustainable Nash equilibrium depends on the objective functional properties besides the rate of discount. In an asymmetric game, the sustainability depends on how the number of actors are divided into groups harvesting population age classes in different locations. The collectively optimal and Nash equilibria are shown to be globally asymptotically stable for optimal feedback solutions. Stochastic recruitment makes harvesting more conservative in both the optimal solution and various Nash equilibria.

Keywords: fisheries, age-structured models, differential games, Markov-perfect Nash equilibria

JEL Classification: Q2

1 Introduction

From its very beginning the economics of renewable biological resources has specified harvested populations as homogenous biomass (Gordon 1954) while the related biological research has developed a life history theory (Roff 1992). This theory describes individuals by their birth, growth, reproduction and death and is directly linked with quantitative genetics and natural selection. It assumes that natural selection operates via life history traits (age of maturation, number of offspring, energy allocation, etc) and maximizes some measure of fitness[1]. At the population level, the theory has led to age-, size and stage-structured models that have been estimated to numerous plant, mammal and fish species (Caswell 2001). Resource economics has an open avenue to proceed from biomass harvesting toward describing the management of biological populations by advanced models with strong theoretical and empirical bases and numerous application possibilities. Our study aims to proceed solving one obstacle in this development by presenting a model with any number of age classes that allows to obtain closed form solutions.

Hannesson (1975), Clark (1976), Reed (1980) and Getz (1980) developed the first generation of economic models on age-structured systems but present somewhat restrictive pulse fishing or maximum sustainable yield solutions. A new generation of models include more realistic age-structured fishery models (Diekert et al. 2010b), size-structured forestry models (Assmuth et al. 2017) and age/sex-structured models for mammals (Pekkarinen et al. 2014). With only few exceptions, these studies apply numerical optimization and analytical results, not to mention closed form solutions, are scarce or nonexistent. In addition to results for collectively optimal solutions this holds especially to game theoretical equilibria for which only numerical open loop solutions have been presented (Diekert et al.

[1] Interestingly, the life history theory applies economic concepts such as investments, discounting, capital, effort, trade-offs and methods like optimal control and dynamic programming (Roff 1992, Samuelson 1977).

2010a). Thus, we contribute to this literature by presenting analytical and closed form solutions for both the optimal and game theoretical setting.

Closed form solutions for the biomass model have been developed within the Fish War literature. Levhari and Mirman (1980) show that assuming logarithmic utility and a specific growth function enables to solve optimal and Markov-perfect Nash equilibria. Fisher and Mirman (1992, 1996) extends the analysis to two species and Doyen et al. (2018) to any number of species. One feature of the model in these studies is that the population intrinsic growth rate is infinite implying that interior steady state always exist. This feature is relaxed in Antoniadou et al (2013) and Mitra and Sorger (2014) and the later authors show that in the Markov-perfect Nash equilibrium the population survives only if the rate of discount multiplied by the number of actors remains below the intrinsic growth rate.

"Optimal" extinction is a much discussed and somewhat controversial issue in the economics of renewable resources and we recall that it is originally given as a warning of the possible consequences of sole ownership (Clark (1973, cf. Begon et al 2006, p. 459). Clearly, this question becomes much more actual in "tragedy of the commons" context but besides the contribution by Mitra and Sorger (2014) results for nonlinear models and finite number of actors have been lacking (cf. Clark 1990, p 155-). Our analysis shows explicitly how, in the age-structured model, the existence of optimal sustainable solutions depends on fundamental biological factors like the number of spawers produced by one spawner, i.e. the reproductive rate, and on the internal delays in the age-structured system. It turns out that in the age-structured models the existence of optimal sustainable solutions is much more critical compared to the biomass model. In our game theoretical setups we obtain new results on this question showing how the existence depends on the objective function parameters beyond the rate of interest.

Our study is an extension of Quaas and Tahvonen (2018) where the model is analyzed assuming two age classes. In addition to extending the model to include any number of age classes we study a symmetrical game theoretical equilibrium where all actors harvest two age classes, derive new results for the existence of sustainable solutions, show global asymptotic stability of steady states and extend the analysis to stochastic recruitment. We first present the model and assumptions on functional forms. We solve the collectively optimal solution in closed form applying dynamic programming and prove the existence and global asymptotic stability of sustainable solutions. Our first game theoretical setup assumes that all actors are perfectly similar and harvest either one or two age classes. We find conditions for the existence of sustainable equilibria and global steady state stability. Next, an asymmetric Markov-perfect Nash equilibrium is solved given the group of actors harvesting the young age class is different from the group harvesting the old and that the groups may differ in their numbers and a utility function parameter. Finally, we show that stochastic recruitment makes harvesting more conservative in all the equilibria studied. With some exceptions, the proofs are given in the appendix.

2 Model structure and assumptions

Let $x_{st}, s = 1,2,...,m, t = 0,1,...$ denote the biomass in age class s at the beginning of period t. Assume $m \geq 1$. Recruitment occurs after harvest and depends on population size and structure and is given by a continuous twice differentiable recruitment function f. The survivability of individuals of age class s to age class $s+1$ is $0 < \gamma_s < 1$, where $s = 1,...,m-1$. The age specific per period harvest is $h_{st}, s = 1,...,m, t = 0,1,...$ .The population development over time can now be specified as

$$x_{1,t+1} = f\left(x_{1t},...,x_{mt},h_{1t},...,h_{mt}\right), \quad (1)$$

$$x_{s+1,t+1} = \gamma_s\left(x_{st} - h_{st}\right), s = 1,...,m-1, \quad (2)$$

$$x_{s0} = \hat{x}_s, s = 1,...,n. \quad (3)$$

Additionally, the variables must satisfy

$$h_{st} \geq 0, x_{st} \geq 0, s = 1,...,m, t = 0,1,... \; . \quad (4)$$

The per periodic utility from harvesting depends on total age class specific harvest and is given by a continuous twice differentiable utility function U. Denoting the per period discount factor by $\rho = 1/(1+r)$, where r is the rate of discount the problem is to

$$\max_{\{h_{st},s=1,...,m,t=0,1,...\}} \sum_{t=0}^{\infty} U(h_{1t},...,h_{mt})\rho^t, \tag{5}$$

subject to (1)-(4). As such this specification is essentially similar with those in Reed (1980) and Getz (1980) which, however, restrict the analysis to maximum sustainable yield steady states.

To obtain an analytically solvable specification assume that the population is harvested from two age classes k and m, where $1 \le k \le m$. Parameters $0 \le n_k$ and $0 \le n_m$ denote the number of fishers harvesting age classes k and m respectively. Utility is obtained from the two age classes separately and are given as

$$U_i(h_{it}) = n_i u_i \frac{h_{it}^{1-\theta} - 1}{1-\theta}, \; i = k, m, \tag{6}$$

where $\theta > 0$, $u_i \ge 0, i = k, m$ and $h_{it}, i = k, m, t = 0, 1, ...$ denote the harvest of individual (similar) fisherman. When $\theta = 1$, we write $n_i u_i U(h_{it}) = n_i u_i \, ln(h_{it})$. The population recruitment depends on the number of fish in age class m (after harvesting) and is given as

$$f(x_{mt} - n_m h_{mt}) = \alpha\left[\beta(x_{mt} - n_m h_{mt})^{1-\eta} + 1 - \beta\right]^{\frac{1}{1-\eta}}, \tag{7}$$

where $\alpha > 0$, $0 < \beta < 1$ and $\eta \ge 1$. Notice that when $\eta \to 1$, f approaches the growth Cushing (1973) recruitment function $\alpha(x_{mt} - n_m h_{mt})^{\beta}$ but we assume $\eta > 1$, if not stated otherwise. This assumption implies that the slope of the recruitment function at the origin is bounded from above, and that recruitment remains bounded from above even if the spawning stock becomes very large – both consistent with principles of recruitment biology. When $\eta = 2$, equation (7) represents the Beverton and Holt (1957) formulation, i.e.

$$f\left(x_{mt} - n_m h_{mt}\right) = \frac{\hat{\alpha}\left(x_{mt} - n_m h_{mt}\right)}{1 - \hat{\beta}\left(x_{mt} - n_m h_{mt}\right)}, \text{ where } \hat{\alpha} = \alpha\beta^{-1} \text{ and } \hat{\beta} = \left(1-\beta\right)^{-1}.$$

The assumptions that only the oldest age class reproduces and die after reproduction refers to *semelparous* species such as eel and Pacific salmon (Jennings et al 2001). Both of these migratory fish species are harvested from some young age class besides harvesting the oldest just before spawning.

Using $\rho = 1/(1+r)$ to denote the discount factor with discount rate $r > 0$, the collective optimization problem is

$$V\left(\mathbf{x}_0\right) = \max_{\{h_{kt}, h_{mt}, t=0,1,...\}} \sum\nolimits_{t=0}^{\infty} \left[n_k u_k \frac{h_{kt}^{1-\theta} - 1}{1-\theta} + n_m u_m \frac{h_{mt}^{1-\theta} - 1}{1-\theta} \right] \rho^t \tag{8}$$

subject to

$$x_{1,t+1} = \alpha\left[\beta\left(x_{mt} - n_m h_{mt}\right)^{1-\eta} + 1 - \beta\right]^{\frac{1}{1-\eta}}, \tag{9}$$

$$x_{s+1,t+1} = \gamma\left(x_{st} - n_k h_{st}\right), h_{st} = 0 \;\; for\, s \neq k,\, s = 1,...,m-1, \tag{10}$$

$$x_{s0} > 0, s = 1,..., m, \tag{11}$$

$$h_{it} \geq 0, i = k, m, x_{st} \geq 0, s = 1,...,m, t = 0,1,... . \tag{12}$$

Notice that when only the oldest age class is harvested, the model coincides the Deriso (1980) delay difference fishery model. Another special case is obtained when only some age class $1 < k < m$ is harvested. In Clark (1990, p. 197-) the steady state solutions for closely related setups are discussed. Assuming $m = 1$, we obtain a version of the standard biomass model for optimizing the escapement $x_t - h_t$ (Clark 1990, p. 198-).

3 Collectively optimal feedback solution

As in the fish war literature and as in Mitra and Sorger (2014), we set $\theta=\eta$ in order to obtain analytical closed form solutions. We hypothesize that the value function has the form

$$V\left(\mathbf{x}\right)=\phi+\sum_{i=1}^{m}\phi_i\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}$$

where $\phi,\phi_s,s=1,...,m$ are unknown parameters. Thus, we attempt to solve the Bellman equation

$$\phi+\sum_{i=1}^{m}\phi_i\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}=$$
$$\max_{\{h_k,h_m\}}\left\{u_kn_k\frac{h_k^{1-\eta}-1}{1-\eta}+u_mn_m\frac{h_m^{1-\eta}-1}{1-\eta}+\rho\left\{\phi+\phi_1\frac{\alpha^{1-\eta}\left[\beta\left(x_k-n_mh_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{k+1}\frac{\left[\gamma\left(x_k-n_kh_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{m}\phi_i\frac{\left(\gamma x_{i-1}\right)^{1-\eta}-1}{1-\eta}\right\}\right\}.$$

Maximization of the RHS leads to $u_kn_kh_k^{-\eta}-\rho\phi_{k+1}\gamma n_k\left(x_k-n_kh_k\right)^{-\eta}=0$ and $u_mn_mh_m^{-\eta}-\rho\phi_1\alpha^{1-\eta}\beta n_m\left(1-n_mh_m\right)^{-\eta}=0.$

Postulating $h_j=\omega_jx_j,j=k,m$, where $\omega_i,i=k,m$ are constants implies

$$u_k\omega_k^{-\eta}-\rho\phi_{k+1}\gamma\left[\gamma\left(1-n_k\omega_k\right)\right]^{-\eta}=0\Leftrightarrow\phi_{k+1}=\frac{u_k\omega_k^{-\eta}}{\left[\gamma\left(1-n_k\omega_k\right)\right]^{-\eta}\gamma\rho},\tag{13}$$

$$u_m\omega_m^{-\eta}-\rho\phi_1\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{-\eta}=0\Leftrightarrow\phi_1=\frac{u_m\omega_m^{-\eta}}{\alpha^{1-\eta}\left(1-n_m\omega_m\right)^{-\eta}\beta\rho}.\tag{14}$$

Postulating again $h_j=\omega_jx_j,j=k,m$, the Bellman equation reads

$$\begin{aligned}&\phi+\sum_{i=1}^{m}\phi_i\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}-u_kn_k\frac{\left(\omega_kx_k\right)^{1-\eta}-1}{1-\eta}-u_mn_m\frac{\left(\omega_mx_m\right)^{1-\eta}-1}{1-\eta}\\&-\rho\left\{\phi+\phi_1\frac{\alpha^{1-\eta}\left[\beta\left(x_m-n_m\omega_mx_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{k+1}\frac{\left[\gamma\left(x_k-n_k\omega_kx_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{k}\frac{\phi_i\left[\left(\gamma x_{i-1}\right)^{1-\eta}-1\right]}{1-\eta}\right\}=0.\end{aligned}\tag{15}$$

Since (15) must hold with any positive $x_i,i=1,...,k$, we obtain

$$\phi_i-\rho\phi_{i+1}\gamma^{1-\eta}=0,\ i=1,...,k-1,\tag{16}$$

$$\phi_k - u_k n_k \omega_k{}^{1-\eta} - \rho\phi_{k+1}\gamma^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta} = 0, \tag{17}$$

$$\phi_i - \rho\phi_{i+1}\gamma^{1-\eta} = 0,\, i = k+1,\ldots,m-1, \tag{18}$$

$$\phi_m - u_m n_m \omega_m{}^{1-\eta} - \rho\phi_1\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta} = 0. \tag{19}$$

From (16) and (18)

$$\phi_k = \phi_1\gamma^{(k-1)(\eta-1)}\rho^{1-k} = \phi_1\delta_k\,, where\ \delta_k \equiv \gamma^{(k-1)(\eta-1)}\rho^{1-k}\,, \tag{20}$$

$$\phi_m = \phi_{k+1}\gamma^{(m-k-1)(\eta-1)}\rho^{1+k-m} = \phi_{k+1}\delta_m\,, where\, \delta_m \equiv \gamma^{(m-k-1)(\eta-1)}\rho^{1+k-m}. \tag{21}$$

Apply (20) and (21) and write (17) and (19) as

$$\phi_1\delta_k - u_k n_k \omega_k{}^{1-\eta} - \rho\phi_{k+1}\gamma^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta} = 0, \tag{22}$$

$$\phi_{k+1}\delta_m - u_m n_m \omega_m{}^{1-\eta} - \rho\phi_1\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta} = 0. \tag{23}$$

Elimination of $\phi_j,\, j = 1, k+1$ from (22) and (23) by (13) and (14) leads to

$$\frac{u_m\delta_k\omega_m{}^{-\eta}\alpha^{\eta-1}\left(1-n_m\omega_m\right)^{\eta}}{\beta\rho} - u_k\omega_k{}^{-\eta} = 0, \tag{24}$$

$$\frac{u_k\delta_m\omega_k{}^{-\eta}\gamma^{\eta-1}\left(1-n_k\omega_k\right)^{\eta}}{\rho} - u_m\omega_m{}^{-\eta} = 0\,, \tag{25}$$

which, given the assumptions $0<\rho\le 1,\, \eta>1,\, \alpha>0, 0<\beta<1, \gamma>0, k\ge 1,\, m\ge 2, k<m,\, m,k\in\mathbb{N}$, have the solution

$$\omega_k = u_k^{\frac{1}{\eta}}\frac{\left(\delta_k\,\delta_m\right)^{\frac{1}{\eta}} - \left(\rho^2\,\alpha^{1-\eta}\,\beta\gamma^{1-\eta}\right)^{\frac{1}{\eta}}}{n_k\left(\delta_k\,\delta_m\,u_k\right)^{\frac{1}{\eta}} + n_m\left(\delta_k\,u_m\,\rho\gamma^{1-\eta}\right)^{\frac{1}{\eta}}} \tag{26}$$

$$\omega_m = u_m^{\frac{1}{\eta}}\frac{\left(\delta_k\,\delta_m\right)^{\frac{1}{\eta}} - \left(\rho^2\,\alpha^{1-\eta}\,\beta\gamma^{1-\eta}\right)^{\frac{1}{\eta}}}{n_k\left(\delta_m u_k\,\rho\alpha^{1-\eta}\,\beta\right)^{\frac{1}{\eta}} + n_m\left(\delta_k\,\delta_m u_m\right)^{\frac{1}{\eta}}} \tag{27}$$

Proposition 1. Assuming

$$\rho^{-m} < \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}, \tag{28}$$

the steady state exists, is unique and

$$x_1 = \left(\frac{1-\beta}{\alpha^{\eta-1} - \beta\lambda^{1-\eta}}\right)^{\frac{1}{1-\eta}},\ \lambda = \gamma^{m-1}(1 - n_k\omega_k)(1 - n_m\omega_m) \tag{29}$$

$$x_s = \gamma^{(s-1)}(1 - n_k\omega_k)x_{s-1},\ \omega_k \equiv 0 \text{ for } s < k+1, s = 2,...,m, \tag{30}$$

$$\textit{where given}(26)\textit{and}(27)\ \lambda = \gamma^{m-1}\rho^{2/n}(\alpha\gamma)^{(1-n)/n}\left(\frac{\beta}{\delta_k\delta_m}\right).$$

Proof, Appendix A.

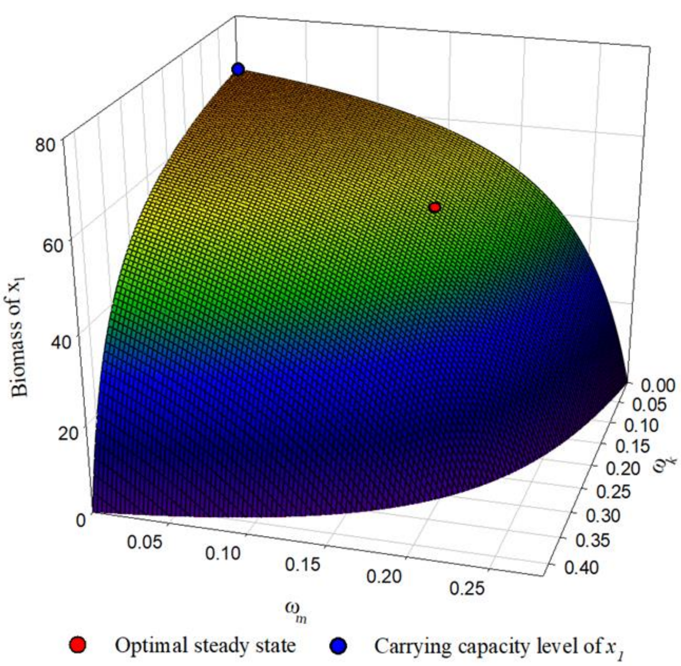

Figure 1. Size of age class x_1 as a function of ω_m and ω_k and the optimal steady state.

Note: $\eta = 2, \alpha = 8, \beta = \gamma = 0.9, \rho = 0.99, k = 2, m = 4, u_k = 1.5, u_m = 1.2, n_k = 2, n_m = 3$

An example of the dependence of the steady on ω_k and ω_m is shown in Figure 1. To interpret the existence condition (28) notice that $\alpha\beta^{\frac{1}{1-\eta}}$ is the derivative of the recruitment function (7) when the number of oldest fish net of harvest approaches zero. For the biomass model $m=1$ and we obtain the familiar condition $r<\alpha\beta^{\frac{1}{1-\eta}}-1$ (cf. Clark 1990, p. 231). Multiplying $\alpha\beta^{\frac{1}{1-\eta}}$ by the survivability factor γ^{m-1} yields the maximum reproductive rate, i.e. number of spawners produced by each spawner in the absence of density dependence in recruitment. According to Myers et al (1999) this is perhaps the most fundamental parameter in population biology and for fish species it typically varies between 1-7. The maximum reproductive rate must exceed one for the population to be biologically viable. Writing (28) as

$$1<\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}\rho^{m},$$

shows that the economically discounted maximum reproductive rate must be higher than one for the population be economically viable i.e. for the optimal sustainable harvesting solution to exist. The critical discount factor is higher (and interest rate lower), the lower are α and γ and the higher are β,η and the number of maturation periods m. For example assuming $\alpha=6, \beta=0.9, \gamma=0.8$, η=2 and $m=4$, the maximum reproductive rate equals 3.41 and the critical discount rate 36%. Decreasing the level of α to 2 decreases maximum reproductive rate to 1.4 and the critical discount rate to 3.3%.

Proposition 2. Given the existence of optimal positive steady state population level, the steady state is globally and asymptotically stable for the optimal feedback solutions.

Proof, Appendix A.

The proof utilizes the fact that the dynamic system (9)-(10) with feedback harvesting rules can be transformed to a linear difference equation system. Given condition (28), it then follows that the unique steady state of this system is globally stable for optimal feedback solutions.

Proposition 3. Given $\rho^{-m} < \alpha\gamma^{m-1}\beta^{\frac{1}{1-\eta}}$ *the optimal steady state satisfies the properties:*

a) $\frac{\partial h_k}{\partial u_k} > 0,\ \frac{\partial h_m}{\partial u_k} < 0,\ \frac{\partial x_s}{\partial u_k} = 0 \text{ for } s = 1,...,k,\ \frac{\partial x_s}{\partial u_k} < 0 \text{ for } s = k+1,...,m,\ \frac{\partial\left(\sum_{s=1}^{m} x_s\right)}{\partial u_k} < 0,$

$$\frac{\partial h_k}{\partial u_m} < 0,\ \frac{\partial h_m}{\partial u_m} > 0,\ \frac{\partial x_s}{\partial u_m} = 0 \text{ for } s = 1,...,k,\ \frac{\partial x_s}{\partial u_m} > 0 \text{ for } s = k+1,...,m,\ \frac{\partial\left(\sum_{s=1}^{m} x_s\right)}{\partial u_m} > 0,$$

b) $\frac{\partial x_s}{\partial n_k} = 0 \text{ for } s = 1,...,k,\ \frac{\partial x_s}{\partial n_k} < 0 \text{ for } s = k+1,...,m,\ \frac{\partial\left(\sum_{s=1}^{m} x_s\right)}{\partial n_k} < 0,\ \frac{\partial h_k}{\partial n_k} < 0,\ \frac{\partial(n_k h_k)}{\partial n_k} > 0,\ \frac{\partial h_m}{\partial n_k} < 0,$

$$\frac{\partial h_m}{\partial n_m} < 0,\ \frac{\partial(n_m h_m)}{\partial n_m} > 0,\ \frac{\partial h_k}{\partial n_m} < 0,\ \frac{\partial x_s}{\partial n_m} = 0 \text{ for } s = 1,...,k,\ \frac{\partial x_s}{\partial n_m} > 0 \text{ for } s = k+1,...,m,\ \frac{\partial\left(\sum_{s=1}^{m} x_s\right)}{\partial n_m} > 0,$$

c) Given $n_k = n_m = n$, it follows that $\frac{\partial(n\omega_k)}{\partial n} = 0,\ \frac{\partial(n\omega_m)}{\partial n} = 0,\ \frac{\partial x_s}{\partial n} = 0,\ s = 1,...,m$,

d) $\frac{\partial x_s}{\partial \rho} > 0,\ s = 1,...,m,\ \frac{\partial h_m}{\partial \rho} > 0,\ \frac{u_m n_m}{u_k n_k}$ *high enough* $\Rightarrow \frac{\partial h_k}{\partial \rho} < 0,\ \frac{u_m n_m}{u_k n_k}$ *low enough* $\Rightarrow \frac{\partial h_k}{\partial \rho} > 0.$

Proof, Appendix A.

For comparison, recall that in the generic biomass harvesting model the only parameter influencing optimal steady state is the discount factor. Proposition 3 shows that here the optimal population steady state depends not only on discounting, but other properties of the objective function as well. Higher marginal value of fish from older age class increases their harvest level, decreases harvest from the young age class and increases total population size. Higher marginal value of young fish has opposite effects. The number of fishermen harvesting the old and young age classes have similar effects albeit in the case of equal number of fisherman harvesting both age classes their total number does not have any effects. Higher discount factor implies higher population size and higher harvest from the old age class but the harvest from the young age class may increase or decrease.

4 A symmetric Markov perfect-Nash equilibrium: all actors harvest both age classes

Assume that all fisherman harvest both age classes and denote their total number as $n\left(=n_k=n_m\right)$. Thus, each fishermen aims to solve the problem

$$V\left(\mathbf{x}_0\right)=\max_{\{h_{kt},h_{mt},t=0,1,\ldots\}}\sum_{t=0}^{\infty}\left[u_k\frac{h_{kt}^{1-\eta}-1}{1-\eta}+u_m\frac{h_{mt}^{1-\eta}-1}{1-\eta}\right]\rho^t$$

subject to

$$x_{1,t+1}=\alpha\left[\beta\left(x_{mt}-H_{mt}-h_{mt}\right)^{1-\eta}+1-\beta\right]^{\frac{1}{1-\eta}},$$

$$x_{s+1,t+1}=\gamma\left(x_{st}-H_{kt}-h_{st}\right),H_{kt}=h_{st}=0\ \ for\, s\neq k,\, s=1,\ldots,m-1,$$

including (11) and (12) and where $H_{it},i=k,m$ denote the harvest of all other (similar) fisherman. We will analyze only the Markov-perfect Nash equilibria, or Nash equilibria for short and apply Dockner et al (2000, p. 92- and Sorger 2015, 210-). The Bellman equation for each player is written as

$$\phi+\sum_{i=1}^{m}\phi_i\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}=$$
$$\max_{\{h_k,h_m\}}\left\{u_k\frac{h_k^{1-\eta}-1}{1-\eta}+u_m\frac{h_m^{1-\eta}-1}{1-\eta}+\rho\left\{\phi+\phi_1\frac{\alpha^{1-\eta}\left[\beta\left(x_m-H_m-h_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{k+1}\frac{\left[\gamma\left(x_k-H_k-h_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{m}\phi_i\frac{\left(\gamma x_{i-1}\right)^{1-\eta}-1}{1-\eta}\right\}\right\}.$$

Maximization of the RHS implies $u_kh_k^{-\eta}-\gamma\rho\phi_{k+1}\left[\gamma\left(x_k-h_k-H_k\right)\right]^{-n}=0,\ u_mh_m^{-\eta}-\beta\rho\phi_1\alpha^{1-\eta}\left(x_m-H_m-h_m\right)^{-\eta}=0.$

Given $h_i=\omega_ix_i,i=k,m$ and $H_i=\left(n-1\right)h_i$, we obtain

$$\phi_{k+1} = \frac{u_k \omega_k^{-\eta}}{\left[\gamma\left(1-n\omega_k\right)\right]^{-\eta} \gamma\rho}, \tag{31}$$

$$\phi_1 = \frac{u_m \omega_m^{-\eta}}{\alpha^{1-\eta}\left(1-n\omega_m\right)^{-\eta} \beta\rho}. \tag{32}$$

Applying again $h_i = \omega_i x_i, i = k, m$ and $H_i = (n-1)h_i$, the Bellman equation reads as

$$\begin{aligned} &\phi + \sum_{i=1}^{m} \phi_i \frac{\left(x_i^{1-\eta}-1\right)}{1-\eta} - u_k \frac{\left(\omega_k x_k\right)^{1-\eta}-1}{1-\eta} - u_m \frac{\left(\omega_m x_m\right)^{1-\eta}-1}{1-\eta} \\ &-\rho\left\{\phi + \phi_1 \frac{\alpha^{1-\eta}\left[\beta\left(x_m - n\omega_m x_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta} + \phi_{k+1}\frac{\left[\gamma\left(x_k - n\omega_k x_k\right)\right]^{1-\eta}-1}{1-\eta} + \sum_{\substack{i=2\\ i\neq k+1}}^{m} \frac{\phi_i\left[\left(\gamma x_{i-1}\right)^{1-\eta}-1\right]}{1-\eta}\right\} = 0. \end{aligned}$$

Since this equation must hold with any positive $x_s, s = 1,\ldots,k,$ we obtain

$$\phi_i - \rho\phi_{i+1}\gamma^{1-\eta} = 0, i = 1,\ldots,k-1, \tag{33}$$

$$\phi_k - u_k\omega_k^{1-\eta} - \rho\phi_{k+1}\gamma^{1-\eta}\left(1-n\omega_k\right)^{1-\eta} = 0, \tag{34}$$

$$\phi_i - \rho\phi_{i+1}\gamma^{1-\eta} = 0, i = k+1,\ldots,m-1, \tag{35}$$

$$\phi_m - u_m\omega_m^{1-\eta} - \rho\phi_1\alpha^{1-\eta}\beta\left(1-n\omega_m\right)^{1-\eta} = 0. \tag{36}$$

From (34) and (35)

$$\phi_k = \phi_1\delta_k, \text{where } \delta_k = \gamma^{(k-1)(\eta-1)}\rho^{1-k} \tag{37}$$

$$\phi_m = \phi_{k+1}\delta_m, \text{ where } \delta_m = \gamma^{(m-k-1)(\eta-1)}\rho^{1+k-m}. \tag{38}$$

Apply (37) and (38) and write (34) and (36) as

$$\phi_1\delta_k - u_k\omega_k^{1-\eta} - \rho\phi_{k+1}\gamma^{1-\eta}\left(1-n\omega_k\right)^{1-\eta} = 0, \tag{39}$$

$$\phi_{k+1}\delta_m - u_m\omega_m^{1-\eta} - \rho\phi_1\alpha^{1-\eta}\beta\left(1-n\omega_m\right)^{1-\eta} = 0. \tag{40}$$

Elimination of $\phi_i, i=1,k+1$ from (39) and (40) by (31) and (32) yields

$$g_1\left(\omega_m,\omega_k\right) \equiv \frac{u_m\delta_k\omega_m^{-\eta}\alpha^{\eta-1}\left(1-n\omega_m\right)^{\eta}}{\beta\rho} - u_k\omega_k^{-\eta}\left(1+\omega_k-n\omega_k\right) = 0, \tag{41}$$

$$g_2\left(\omega_m,\omega_k\right) \equiv \frac{u_k\delta_m\omega_k^{-\eta}\gamma^{\eta-1}\left(1-n\omega_k\right)^{\eta}}{\rho} - u_m\omega_m^{-\eta}\left(1+\omega_m-n\omega_m\right) = 0. \tag{42}$$

When $\eta>1$, the pair ω_k,ω_m cannot be solved explicitly from (41) and (42) but to proceed solve u_k/u_m from (41) and eliminate it from (42) and obtain

$$y_1\left(\omega_k,\omega_m\right) \equiv \frac{\left[1-\omega_k\left(n-1\right)\right]\left[1-\omega_m\left(n-1\right)\right]}{\left[\left(1-n\omega_m\right)\left(1-n\omega_k\right)\right]^{\eta}} - \left(\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}\right)^{\eta-1}\rho^{-m} = 0. \tag{43}$$

Proposition 4. Assuming $1<\alpha\beta^{\frac{1}{1-n}}\gamma^{m-1}$*, the solution for (41) and (42) exists and is unique.*

Proof: From $g_2\left(\omega_m,\omega_k\right)=0$ or (42) we obtain

$$\omega_k = \frac{\omega_m\gamma^{(k+m\eta)/\eta}\rho^{k/\eta}u_k^{1/\eta}}{\gamma^{(k\eta+m)/\eta}\rho^{m/\eta}\left[u_m\left(1+\omega_m-n\omega_m\right)\right]^{1/\eta} + n\omega_m\gamma^{(k+m\eta)/\eta}\rho^{k/\eta}u_k^{1/\eta}}. \tag{44}$$

Since $\partial\omega_k/\partial\omega_m>0$ and $\omega_k\to 0$ as $\omega_m\to 0$, equation $g_2\left(\omega_m,\omega_k\right)=0$ or (44) defines ω_k as an increasing function of ω_m that starts from the origin. Given $1<\alpha\beta^{\frac{1}{1-n}}\gamma^{m-1}$, equation $y_1\left(\omega_k,\omega_m\right)=0$ (or 43) has the properties $y_1\left(0,0\right)<0$, $\partial y_1\left(\omega_k,0\right)/\partial\omega_k>0$ and $y_1\left(\omega_k,0\right)\to\infty$ as $\omega_k\to n^{-1}$. Additionally, $y_1\left(\omega_k,\omega_m\right)=0$ defines ω_k as a decreasing function of ω_m. Thus, a unique pair ω_k,ω_m solving (43) and (42) exists and this pair must solve (41) as well.□

The existence of solution for (41) and (42) still leaves open the question whether the population survives, i.e. whether the steady state population level is strictly positive. The necessary and sufficient condition for the existence of steady states with $x_s > 0, s = 1,...,m$ is

$$y_2(\omega_k, \omega_m) \equiv (1 - n_k \omega_k)(1 - n_m \omega_m) - \alpha^{-1} \beta^{1/(\eta-1)} \gamma^{1-m} > 0 \tag{45}$$

(A8, Appendix A), where $n_k = n_m = n$ when all actors harvest both age classes. Given $1 < \alpha \beta^{\frac{1}{1-\eta}} \gamma^{m-1}$ both equations $y_i(\omega_k, \omega_m) = 0, i = 1,2$ define ω_k as decreasing functions of ω_m and any solution pair of $y_1(\omega_k, \omega_m) = 0$ with the property $x_s > 0, s = 1,...,m$ must exist below the locus of $y_2(\omega_k, \omega_m) = 0$. Observe that $y_i(\omega_k, \omega_m) = 0, i = 1,2$ are symmetric functions of $\omega_i, i = k, m$. This structure enables us to analyze the existence of steady states with positive population levels. We call an equilibrium positive if it satisfies $x_s > 0, s = 1,...,m$.

Proposition 5. Given $u_k = 0$ *or* $u_m = 0$, *the necessary and sufficient condition for positive Nash equilibrium steady states is*

$$n(\rho^{-m} - 1) < \alpha \beta^{\frac{1}{1-\eta}} \gamma^{m-1} - 1. \tag{46}$$

Proof, Appendix A.

With one actor $(n = 1)$, equation (46) coincide (28) and when $m = 1$, it can be written in the form $nr < \alpha \beta^{\frac{1}{1-\eta}} - 1$, i.e. it coincides the result obtained in Mitra and Sorger (2014) for the biomass model. Given our earlier example with $\alpha = 6, \beta = 0.9, \gamma = 0.8, \eta = 2, m = 4$ the critical interest rate decreases from 36% to 5.6% when n increases from 1 to 10. When $\alpha = 2$ the decrease is from 3.3% to 0.34% showing the sensitivity of positive Nash equilibrium steady state on the number of actors.

Proposition 6. Given $\eta > 1, u_k \geq 0, u_m \geq 0$, *the condition*

$$n\left(\rho^{-m} - 1\right) > \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1} - 1 \tag{47}$$

is sufficient for the nonexistence of positive Nash equilibrium steady states.

Proof, Appendix A.

Proposition 7. Given $\eta > 1, u_k > 0, u_m > 0$, *the condition*

$$n\left(\sqrt{\rho^{-m}} - 1\right) < \sqrt{\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}} - 1 \tag{48}$$

is sufficient for the existence of a positive Nash equilibrium steady state.

Proof, Appendix A.

Let us denote the level of ρ satisfying (47) as equality by ρ_1 and the level of ρ satisfying (48) by ρ_2, respectively.

Corollary 1: At the Nash equilibrium the pair ω_k, ω_m *is higher (lower) the higher is* u_k / u_m. *Higher value of k decreases* ω_k *and increases* ω_m *if* $\rho\gamma^{1-n} - 1 < 0$ *and vice versa. Given* $\rho_1 < \rho < \rho_2$ *the existence of positive Nash equilibrium steady state depends on* u_k / u_m *and* k.

Proof, Appendix A.

Propositions 5-7 are illustrated in Figure 2 where the upper solid line shows ω_k as a function of ω_m as defined by (43). This function coincides with the boundary of positive steady states (dashed line) when either $\omega_k = 0$ or $\omega_m = 0$ as proved in Proposition 5. This implies condition (46) as a necessary and sufficient condition for positive Nash feedback equilibrium steady states given only the young or old age class is harvested. When both age classes are harvested condition (47) implies that no positive Nash feedback equilibrium exists (Proposition 6). When the locus of (43) exists entirely below the existence boundary of positive steady states (45), positive steady states always exists independently on u_m, u_k and k . This outcome follows under condition (48). When both (47) and (48) are violated the pair ω_m, ω_k satisfying (41 and (42) lies between the solid lines and then the existence of positive Nash equilibrium depends on u_m, u_k and k as shown in Proposition 7.

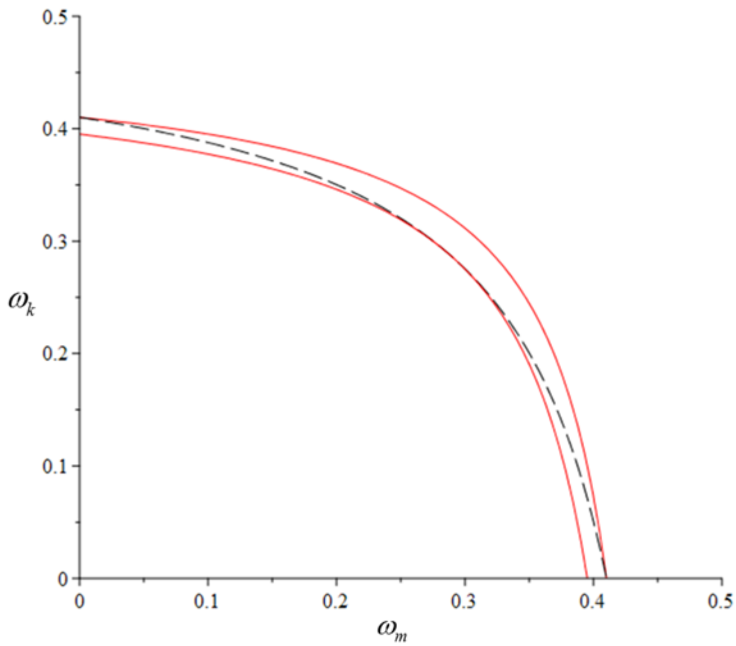

Figure 2. Illustration of Propositions 6,7 and 8.
Note: Upper solid line: Propositions 6 and 7, Lower solid line: Proposition 8
Dashed line: The boundary for positive steady states
$\alpha = 4, \beta = 0.93, \gamma = 0.95, m = 4, n = 2, \eta = 1.15, \rho \approx 0.743$ upper solid line
$\rho \approx 0.772$ lower solid line

The role of discounting and number of players in Propositions (5)-(7) and Corollary 1 are illustrated in Figure 3. A combination of N and r above the dashed line is sufficient for the nonexistence of positive Nash equilibrium steady state. Accordingly, a combination existing below the solid line is sufficient for the positive steady state. The two dots are examples of critical number of players/interest rate combinations and increasing either of these two parameters imply that no positive steady state exist. The number of players have strong effect on the existence of positive steady state. Given the parameters in Figure 4 and one actor the critical interest rate is 46% while with one hundred actors the value is ca. 1%.

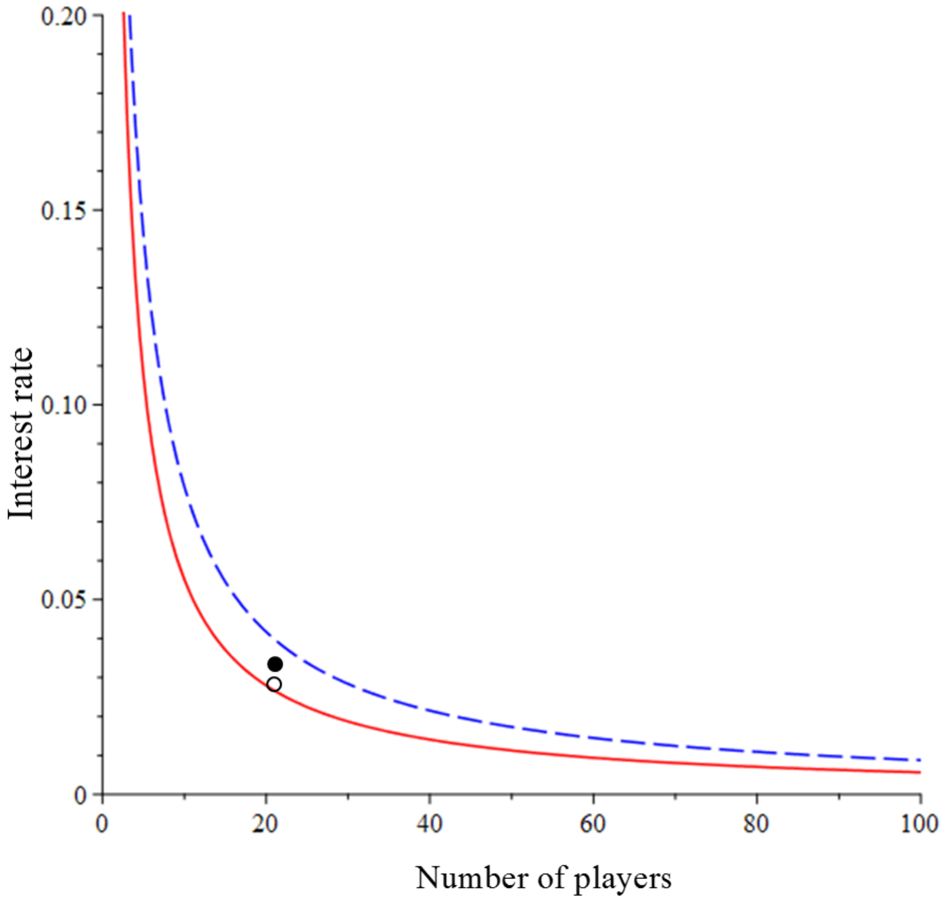

Figure 3. Illustration of Propositions 5-8 and Corollary 2.
Note: $\alpha = 8, \beta = 0.9, \gamma = 0.8, m = 4, n = 2$
Solid line: Below the line positive steady states exist (equation 48)
Dashed line: Above the line no positive steady states exist (equation 47)
Circle: Critical interest rate when $n = 20, u_m = 2, u_k = 1$
Open circle: Critical interest rate when $n = 20, u_m = 1, u_k = 4$
Given N=1, the critical interest rate is 0.46.

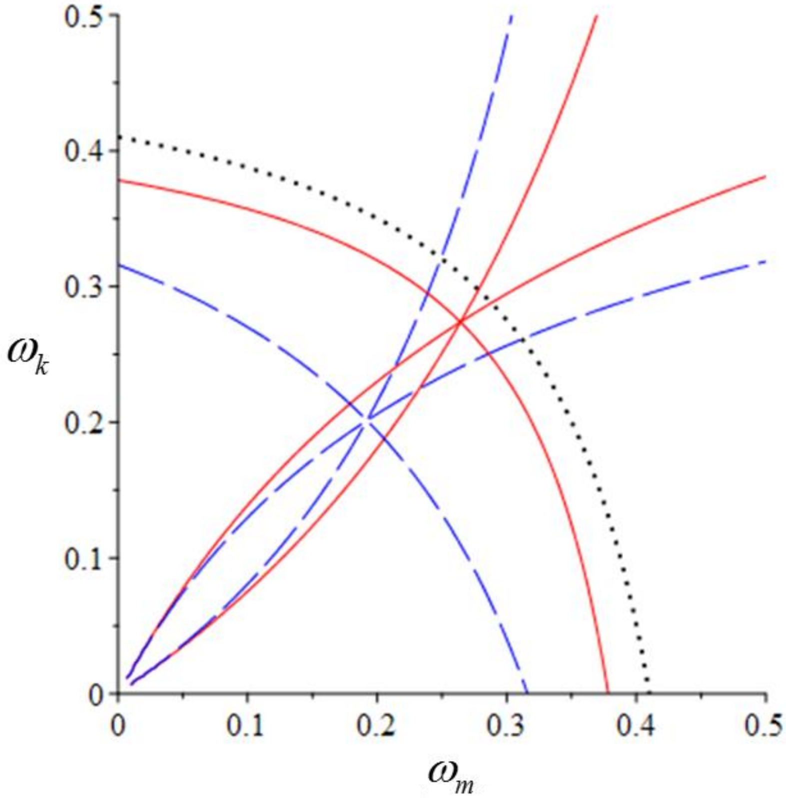

Figure 4. Illustration of Proposition 8.

Note: Dotted line: Boundary for positive population levels

Solid line: Nash equilibrium, Dashed line: Optimal steady state

$\alpha = 4, \beta = 0.93, \gamma = 0.95, m = 4, n = 2, \eta = 1.15, \rho = 0.8, u_m = u_k = 1.$

With zero rate of interest the conditions (46-48) reduce to the condition of population biological viability $1 < \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}$, i.e. number of actors has a role on population survivability only when interest rate is positive. In spite of this the equilibrium steady state depends on the number of actors even with zero rate of interest. Given $\rho = \eta = 1$, the system (41) and (42) has the solution

$$\omega_k = \frac{u_k(1-\beta)}{nu_k(1-\beta) + \beta u_k + u_m}, \; \omega_m = \frac{u_m(1-\beta)}{nu_m(1-\beta) + \beta(u_k + u_m)},$$

implying that $n\omega_k$ and $n\omega_m$ increase in the number of actors and that the Nash equilibrium population level is lower the higher is the number of actors.

We next compare the Nash and collectively optimal outcomes. For this purpose let ω_{ko} and ω_{mo} refer to collectively optimal outcome and ω_{kn} and ω_{mn} to the Nash equilibrium respectively.

Proposition 8. Given $\eta > 1$, $n_k = n_m = n$, it follows that $\omega_{io} < \omega_{in}, i = k, m$.

Proof, Appendix A.

Thus, we obtained the expected result that in the Nash equilibrium harvesting is less conservative compared to the collective equilibrium. Proposition 8 is illustrated in Figure 4.

Corollary 2: When the positive Nash equilibrium exists, it is globally asymptotically stable.

Proof: The global asymptotical stability follows from the proof of Proposition 2. □

5. Asymmetric Markov-perfect Nash equilibrium: two separate groups of harvesters

Given separate groups of actors harvesting age classes k and m $(1 \le k < m)$ the Bellman equations are written as

$$\phi_{10} + \sum_{i=1}^{m} \phi_{1i} \frac{\left(x_i^{1-\eta} - 1\right)}{1-\eta} = \max_{\{h_k\}} \left\{ u_k \frac{h_k^{1-\eta} - 1}{1-\eta} + \rho \left\{ \phi_{10} + \phi_{11} \frac{\alpha^{1-\eta}\left[\beta\left(x_m - n_m h_m\right)^{1-\eta} + 1 - \beta\right] - 1}{1-\eta} + \phi_{1,k+1} \frac{\left[\gamma\left(x_k - H_k - h_k\right)\right]^{1-\eta} - 1}{1-\eta} + \sum_{\substack{i=2 \\ i \neq k+1}}^{m} \phi_{1i} \frac{\left(\gamma x_{i-1}\right)^{1-\eta} - 1}{1-\eta} \right\} \right\}, \quad (49)$$

and

$$\phi_{20}+\sum_{i=1}^{m}\phi_{2i}\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}=$$
$$\max_{\{h_m\}}\left\{u_m\frac{h_m^{1-\eta}-1}{1-\eta}+\rho\left\{\phi_{20}+\phi_{21}\frac{\alpha^{1-\eta}\left[\beta\left(x_m-H_m-h_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{2k+1}\frac{\left[\gamma\left(x_k-n_kh_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{m}\phi_{2i}\frac{\left(\gamma x_{i-1}\right)^{1-\eta}-1}{1-\eta}\right\}\right\}, \quad (50)$$

respectively. By maximizing the RHS of both equations and applying $H_i=\left(n_i-1\right)h_i, h_i=\omega_i x_i, i=k,m$ we obtain

$$\phi_{1,k+1}=\frac{u_k\omega_k^{-\eta}}{\gamma\rho\left[\gamma\left(1-n_k\omega_k\right)\right]^{-\eta}}, \quad (51)$$

$$\phi_{21}=\frac{u_m\omega_m^{-\eta}}{\rho\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{-\eta}}. \quad (52)$$

Applying again $h_i=\omega_i x_i, i=k,m$ the Bellman equation for harvesters of age class *k* reads as

$$\phi_{10}+\sum_{i=1}^{m}\phi_{1i}\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}-u_k\frac{\left(\omega_k x_k\right)^{1-\eta}-1}{1-\eta}$$
$$-\rho\left\{\phi_{10}+\phi_{11}\frac{\alpha^{1-\eta}\left[\beta\left(x_m-n_m\omega_m x_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{1,k+1}\frac{\left[\gamma\left(x_k-n_k\omega_k x_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{m}\frac{\phi_{1i}\left[\left(\gamma x_{i-1}\right)^{1-\eta}-1\right]}{1-\eta}\right\}=0.$$

Proceeding as before and after eliminating $\phi_{1i}, i=2,...,k,k+1,...,m$ leads to the conditions

$$\phi_{11}\delta_k-u_k\omega_k^{1-\eta}-\rho\phi_{1,k+1}\gamma^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta}=0, \quad (53)$$

$$\phi_{1,k+1}\delta_m-\rho\phi_{11}\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta}=0. \quad (54)$$

The Bellman equation for the harvesters of age class *m* is given as

$$\phi_{20}+\sum_{i=1}^{m}\phi_{2i}\frac{\left(x_i^{1-\eta}-1\right)}{1-\eta}-u_m\frac{\left(\omega_m x_m\right)^{1-\eta}-1}{1-\eta}$$
$$-\rho\left\{\phi_{20}+\phi_{21}\frac{\alpha^{1-\eta}\left[\beta\left(x_m-n_m\omega_m x_m\right)^{1-\eta}+1-\beta\right]-1}{1-\eta}+\phi_{2,k+1}\frac{\left[\gamma\left(x_k-n_k\omega_k x_k\right)\right]^{1-\eta}-1}{1-\eta}+\sum_{\substack{i=2\\ i\neq k+1}}^{m}\frac{\phi_{2i}\left[\left(\gamma x_{i-1}\right)^{1-\eta}-1\right]}{1-\eta}\right\}=0,$$

implying after eliminating $\phi_{2i}, i=2,...,k,k+2,...,m$ the conditions

$$\phi_{21}\delta_k-\rho\phi_{2,k+1}\gamma^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta}=0, \quad (55)$$

$$\phi_{2,k+1}\delta_m-u_m\omega_m^{1-\eta}-\phi_{21}\rho\alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta}=0. \quad (56)$$

Elimination of $\phi_{i1},\phi_{i,k+1},i=1,2$ from (53)-(56) leads to

$$q_i \equiv \frac{\left(1-n_i\omega_i\right)^{\eta}\left(1-n_j\omega_j\right)^{\eta-1}}{\omega_i\left(n_i-1\right)-1}+\Phi=0, i=k,m, j=k,m, i\neq j, \tag{57}$$

$$\Phi=\beta\alpha^{1-\eta}\gamma^{(1-\eta)(m-1)}\rho^{m}. \tag{58}$$

Elimination of Φ from (57) and (58) implies that any $\omega_i, i=k,m$ solution satisfies

$$\omega_k=\frac{\omega_m}{1+n_k\omega_m-n_m\omega_m}. \tag{59}$$

Proposition 9. Given $\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}>1$, a unique solution $0<\omega_i<n_i^{-1}, i=k,m$ for (57)-(59) exists.

Proof, Appendix A.

To develop conditions implying a positive steady state with $x_s>0, s=1,...,m$ we study solutions satisfying (57)-(59) and the boundary condition for positive steady states (cf. 45)

$$\left(1-n\omega_k\right)\left(1-n\omega_m\right)=\alpha^{-1}\beta^{1/(\eta-1)}\gamma^{1-m}. \tag{60}$$

Applying (59) and (60) enables to obtain

$$\omega_i=-\frac{\gamma^{-m}\left[\sigma+\gamma\beta^{\frac{1}{1-\eta}}\left(n_i-n_j\right)-2\alpha n_i\gamma^{m}\right]}{2\alpha n_i^2}, i=m,k, j=m,k, i\neq j, \tag{61}$$

$$\sigma=\sqrt{\gamma}\beta^{\frac{1}{2(\eta-1)}}\sqrt{\gamma\beta^{\frac{1}{\eta-1}}\left(n_k-n_m\right)^2+4\alpha n_k n_m\gamma^{m}}. \tag{62}$$

Additionally, the solution must satisfy (57), (58), which when specified for $q_k=0$, can be written as

$$\rho^{-m}=\beta\alpha^{(1-\eta)}\gamma^{(1-\eta)(m-1)}\left(1-n_m\omega_m\right)^{1-\eta}\left(1-n_k\omega_k\right)^{-\eta}\left(1+\omega_k-n_k\omega_k\right). \tag{63}$$

Proposition 10. Given $n_k = n_m = n$*, a positive Nash equilibrium steady state satisfying (57), (58) exists iff*

$$n\left(\rho^{-m} - 1\right) < \sqrt{\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}} - 1. \tag{64}$$

The discount factor that solves (64) as an equality is higher than the corresponding discount factor in the condition (28) for the socially optimal solution.

Proof, Appendix A.

According to (64) the number of actors affects the steady state existence only if the discount factor is below one. However, the number of actors has an effect on the steady state independently on discounting. To reveal this assume $\eta = 1$ and apply (57) and (58) to obtain

$$\omega_k = \frac{1-\Phi}{n_k + \Phi(1-n_k)},\ \omega_m = \frac{1-\Phi}{n_m + \Phi(1-n_m)},$$

where $\Phi = \beta$ if $\rho = 1.$ Additionally, both $n_k\omega_k$ and $n_m\omega_m$ increase in number of actors implying that the equilibrium steady state population level decrease with number of actors.

Proposition 11. Given $\eta = 2$*, a positive Nash equilibrium steady state satisfying (57), (58) exists iff*

$$\rho^{-m} < \frac{\sqrt{4\alpha n_k n_m \gamma^m + \beta\gamma\left(n_k - n_m\right)^2}}{2\sqrt{\beta}\sqrt{\gamma}n_k n_m} + \frac{n_k\left(2n_m - 1\right) - n_m}{2n_k n_m}. \tag{65}$$

The lowest discount factor satisfying (65) as an equality is higher than the corresponding discount factor in the case of the socially optimal condition (28).

Proof: The existence claims follow by analogous steps as in the proof of Proposition 10. The discount factor comparison holds true if

$$g(\alpha) \equiv \frac{\sqrt{4\alpha n_k n_m \gamma^m + \beta\gamma(n_k - n_m)^2}}{2\sqrt{\beta}\sqrt{\gamma} n_k n_m} + \frac{n_k(2n_m - 1) - n_m}{2n_k n_m} - \alpha\beta^{-1}\gamma^{m-1} < 0.$$

We obtain $g = 0$ when $1 = \alpha\beta^{-1}\gamma^{m-1}$, $g'(0) < 0, g''(\alpha) < 0$ implying $g(\alpha) < 0$ when $1 < \alpha\beta^{-1}\gamma^{m-1}$. □

Equation (65) does not immediately reveal how the number of actors influence the existence of positive steady state but it can be shown that the RHS approaches one from above given the population is biologically viable and either n_k or n_m approaches infinity. Thus, for any $\rho < 1$, some finite number of actors always implies that the nonexistence of a sustainable equilibrium. Accordingly, without discounting the LHS of (65) equals one implying that the steady state population level remains positive given the population is biologically viable and number of actors is finite. However, we have already observed that without discounting the steady state population level decreases with number of actors. To further clarify how the number of actors affect the steady state in the absence of discounting assume $\rho = 1$, $\eta = 2$, and $n_k = n_m = n$. Equations (57) imply $\omega_k = \omega_m$ and

$$\Phi(n\omega - \omega - 1) - (n\omega - 1)^3 = 0, \tag{66}$$

where $\omega \equiv \omega_k = \omega_m$. When $n \to \infty$, it holds that $\omega \to 0$ for (66) to hold and the value of $n\omega$ that solves (66) approaches $1 - \sqrt{\Phi}$. This implies that in (29) $\lambda \to \frac{\beta}{\alpha}$ and $x_s \to 0$, $s = 1, ..., n$. Thus, with no discounting, the equilibrium population level remains positive but converges toward zero when the number of actors increase without limit. We conjecture that this holds with any $\eta > 1$, and when $n_k \neq n_m$. A numerical example is shown in Figure 5 where $n_k \neq n_m$.

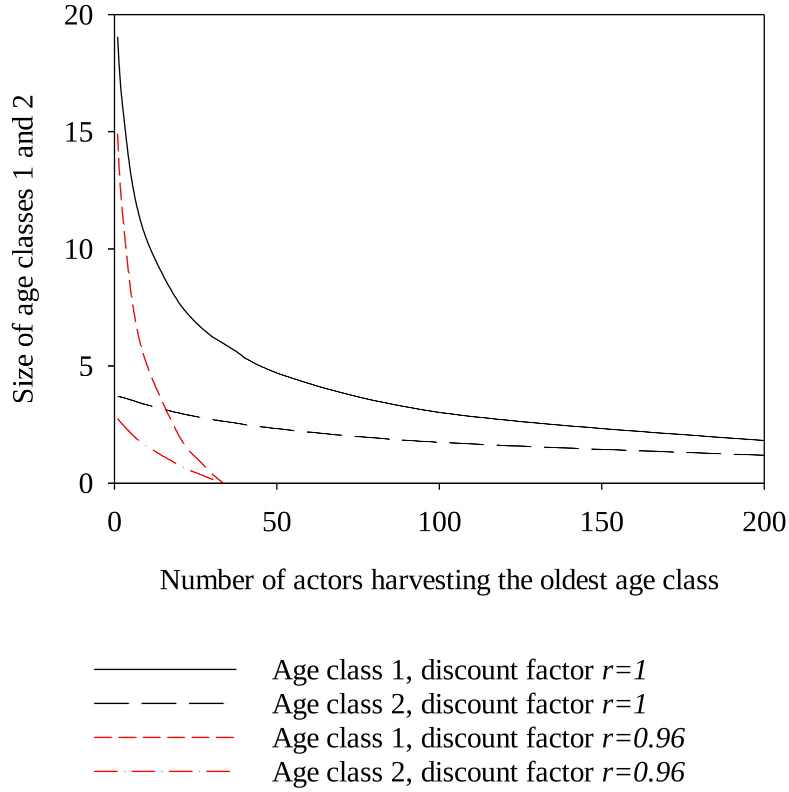

Figure 5. Equilibrium steady state population with and without discounting

Note: $\alpha = 8, \beta = 0.9, \gamma = 0.8, \ m = 2, \eta = 2, n_k = 10$.

Proposition 12. Given $\eta = 2$ *and any choice of* $\alpha > 0, 0 < \beta < 1, 0 < \gamma < 1, 2 < m$ *the discount factor satisfying*

$$\rho^{-m} = \frac{\sqrt{4\alpha n_k n_m \gamma^m + \beta\gamma\left(n_k - n_m\right)^2}}{2\sqrt{\beta}\sqrt{\gamma} n_k n_m} + \frac{n_k\left(2n_m - 1\right) - n_m}{2n_k n_m} \tag{67}$$

is highest when $n_m = n_k$.

Proof: Write $n_k = 1 - n_m$. This implies that the RHS of (66) becomes a concave function of n_m which obtains its minimum value when $n_m = n_k = \frac{1}{2}$. □

Proposition 13. In Nash equilibrium at least ω_k or ω_m *is higher than in the collectively optimal solution satisfying satisfying (24), (25).*

Proof, Appendix A.

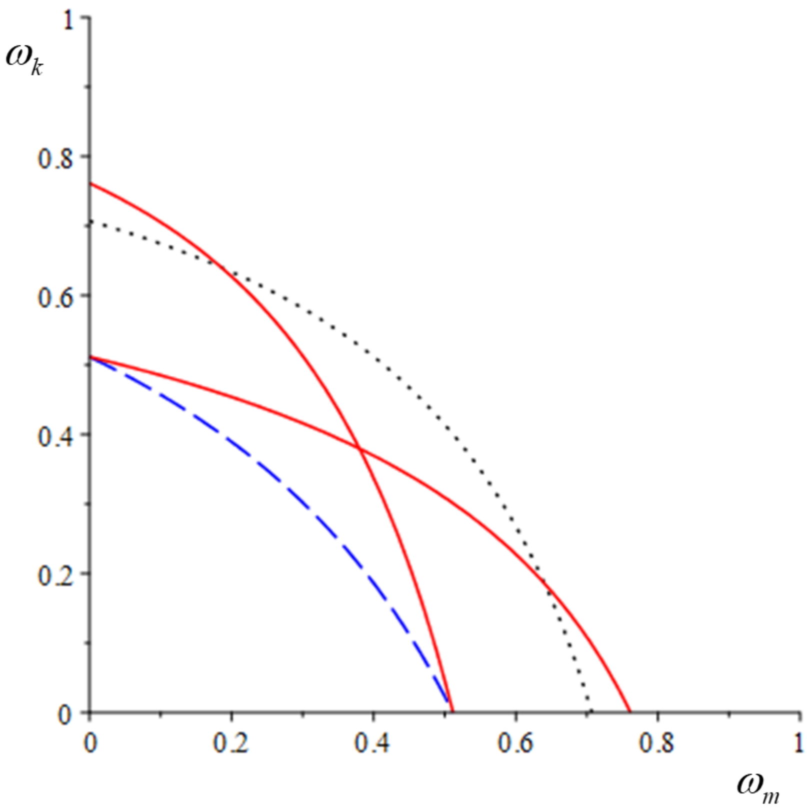

Figure 6. Illustration of Proposition 13
Solid lines: Nash equilibrium
Dashed line: The optimal equilibrium boundary
Dotted line: The boundary for positive steady states
Note: $\alpha = 4, \beta = 0.6, \gamma = 0.8, m = 4, n_k = n_m = 1, \eta = 2, \rho = 0.95.$

Proposition 13 is demonstrated in Figure 6, where $n_k = n_m$. The Nash equilibrium is independent on parameters u_k and u_m while the optimal pair of ω_k, ω_m may exist anywhere on the dotted line. Thus, it is possible that the harvesting of either the young or old age class is more conservative in the Nash equilibrium compared to the optimal outcome. This implies that in contrast to the equilibrium where all actors harvest both age classes it is not possible to rule out an outcome where the Nash equilibrium steady state population is larger compared to the optimal steady state population. This is possible if u_k / u_m is large implying by proposition 3a that the optimal total population steady state level is low while the Nash equilibrium steady state is independent on these parameters.

6 Stochastic recruitment

We assume that recruitment is stochastic and that $\tilde{\alpha}_{t+1}$ follows an iid random process with positive support. The expected value of $\tilde{\alpha}_{t+1}$ equals the value of α in the deterministic setting, i.e. $\alpha = \mathbb{E}\left[\tilde{\alpha}_{t+1}\right]$, to make the deterministic and stochastic models comparable. We guess that the value function has the same form as in the deterministic case. Thus, we attempt to solve the Bellman equation

$$\phi + \sum_{i=1}^{m} \phi_i \frac{\left(x_i^{1-\eta} - 1\right)}{1-\eta} =$$

$$\max_{\{h_k, h_m\}} \left\{ u_k n_k \frac{h_k^{1-\eta} - 1}{1-\eta} + u_m n_m \frac{h_m^{1-\eta} - 1}{1-\eta} + \rho \mathbb{E} \left\{ \phi + \phi_1 \frac{\tilde{\alpha}^{1-\eta}\left[\beta\left(x_k - n_m h_m\right)^{1-\eta} + 1 - \beta\right] - 1}{1-\eta} + \phi_{k+1} \frac{\left[\gamma\left(x_k - n_k h_k\right)\right]^{1-\eta} - 1}{1-\eta} + \sum_{\substack{i=2 \\ i \neq k+1}}^{m} \phi_i \frac{\left(\gamma x_{i-1}\right)^{1-\eta} - 1}{1-\eta} \right\} \right\}.$$

Maximization on the right-hand side leads to

$$u_k n_k h_k^{-\eta} - \rho \phi_{k+1} \gamma^{1-\eta} n_k \left(x_k - n_k h_k\right)^{-\eta} = 0,$$

$$u_m n_m h_m^{-\eta} - \rho \varphi_1 \mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right] \beta n_m \left(x_m - n_m h_m\right)^{-\eta} = 0.$$

Postulating $h_j = \omega_j x_j$, $j = k, m$, and following the same steps in the deterministic case, we obtain

$$\omega_k = \sigma_k^{\frac{1}{\eta}} \frac{(\delta_k \delta_m)^{\frac{1}{\eta}} - \left(\rho^2 \mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]\beta\gamma^{1-\eta}\right)^{\frac{1}{\eta}}}{(\delta_k \delta_m \sigma_k)^{\frac{1}{\eta}} n_k + \left(\delta_k \sigma_m \rho \gamma^{1-\eta}\right)^{\frac{1}{\eta}} n_m},$$

$$\omega_m = \sigma_m^{\frac{1}{\eta}} \frac{(\delta_k \delta_m)^{\frac{1}{\eta}} - \left(\rho^2 \mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]\beta\gamma^{1-\eta}\right)^{\frac{1}{\eta}}}{\left(\delta_m \sigma_k \rho \mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]\beta\right)^{\frac{1}{\eta}} n_k + (\delta_k \delta_m \sigma_m)^{\frac{1}{\eta}} n_m}.$$

For $\eta = 1$, harvest rates equl their deterministic values. For $\eta > 1$, $\mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right] > \left(\mathbb{E}\left[\tilde{\alpha}\right]\right)^{1-\eta} = \alpha^{1-\eta}$, by Jensen's inequality, as $\tilde{\alpha}^{1-\eta}$ is a convex function of $\tilde{\alpha}$. As both ω_k and ω_m decrease in $\mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]$, uncertainty makes optimal harvesting more conservative.

The solutions for the Nash equilibria are very similar as in the stochastic case. The only difference is that (41) and (42) are replaced by

$$g_1(\omega_m, \omega_k) \equiv \frac{u_m \delta_k \omega_m^{-\eta} (1 - n\omega_m)^{\eta}}{\mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]\beta\rho} - u_k \omega_k^{-\eta} (1 + \omega_k - n\omega_k) = 0,$$

$$g_2(\omega_m, \omega_k) \equiv \frac{u_k \delta_m \omega_k^{-\eta} (1 - n\omega_k)^{\eta}}{\rho\gamma^{1-\eta}} - u_m \omega_m^{-\eta} (1 + \omega_m - n\omega_m) = 0.$$

Condition (58) remains very similar, except that we now have

$$\Phi = \mathbb{E}\left[\tilde{\alpha}^{1-\eta}\right]\beta\gamma^{(1-\eta)(m-1)}\rho^m.$$

In both Nash equilibria stochastic recruitment makes harvesting more conservative similarly as in the optimal solution.

6 Conclusions

Our aim has been to present analytical and closed form solutions for harvesting age-structured populations in both optimal and game theoretical settings. Developing economic models on management and harvesting of biological populations to include population internal structure allows to utilize the latest development in population biology and the detailed data available. From the management point of view this development is necessary since according the present knowledge fishing causes age truncation effects which alter population future growth and life history traits. A warning example is the collapse of Atlantic cod stock off Labrador and Newfoundland that belongs to the worst collapses in the history of fisheries and which has been explained to be a course of uncontrolled fishing-induced age and size truncation effects (Olson et al 2004).

Presently, optimization models including population internal structure are analyzed by numerical methods with only handful of exceptions (e.g. Tahvonen 2009). Analytical results make the deductive links between the model assumptions and consequences transparent and are still considered necessary for an economic model to be accepted as a theoretically genuine construction (Lehtinen and Kuorikoski 2007). Analytical and closed form solutions for optimal harvesting models have been developed within the Fish War literature that has this far rest on biomass models for one or several species. Thus, in addition to strengthening the theoretical basis of the existing age-structured fishery models our study offers an extension for the Fish War literature.

We have solved optimal and Markov-perfect Nash equilibria for a specific age-structured model analytically and for several cases in closed form. Similar earlier analysis has not been presented in resource economics. Among other results, we showed that the comparative statics of the steady states of these equilibria differ from the corresponding biomass model. In the symmetric Nash equilibrium where all actors harvest two age classes the harvests are less conservative compared to optimal solutions while in the asymmetric Nash equilibrium with two groups of actors the reverse is possible.

Since Clark (1973) one important and much discussed question has been the existence of optimal sustainable solutions or “optimal extinction”. Most commonly, this has been studied in the sole owner context while a game theoretical many player setup is clearly more relevant. We have presented a set of new results showing how the sustainability depends critically on specific biological factors, number of harvesters and properties of the objective functional beyond the rate of discount. The interplay of the delays in the age-structured system together with the number of actors makes the existence of optimal sustainable solutions more vulnerable that suggested by the biomass models.

Appendix.

Proof of Proposition 1

We study the steady state of the system

$$x_{1,t+1} = \alpha\left[x_{mt}^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta}+1-\beta\right]^{\frac{1}{1-\eta}}, \tag{A1}$$

$$x_{s+1,t+1} = \gamma\, x_{st}\left(1-n_k\omega_s\right),\ \omega_s = 0\ \ for\, s\neq k,\, s=1,...,m-1, \tag{A2}$$

Given $\eta > 1$, this can be transformed to a linear difference equation for $x_{st}^{1-\eta}, s=1,...,m.$ Denote $x_{st}^{1-\eta} \equiv \mathrm{X}_{st}, s=1,...,m.$ Thus,

$$\mathrm{X}_{1,t+1} = \alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta}\mathrm{X}_{mt}+\alpha^{1-\eta}\left(1-\beta\right), \tag{A3}$$

$$\mathrm{X}_{s+1,t+1} = \gamma^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta}\mathrm{X}_{st}, \text{ where } \omega_k = 0\ \ for\, s<k,\, s=1,...,m-1. \tag{A4}$$

Let $\mathrm{X}_s, s=1,...,m$ denote the steady state. Solving (A4) recursively yields $\mathrm{X}_m = \gamma^{(m-1)(1-\eta)}\left(1-n_k\omega_k\right)^{1-\eta}\mathrm{X}_1,$

which enables us to solve X_1 from (A3):

$$\mathrm{X}_1 = \frac{\alpha\gamma^{m\eta+1}\left(1-\beta\right)\left(1-n_k\omega_k\right)^{\eta}\left(1-n_m\omega_m\right)^{\eta}}{\alpha^{\eta}\gamma^{\eta m+1}\left(1-n_m\omega_m\right)^{\eta}\left(1-n_k\omega_k\right)^{\eta}+\alpha\beta\gamma^{m+\eta}\left(1-n_m\omega_m\right)\left(n_k\omega_k-1\right)}, \tag{A5}$$

which after simplification yields (29). Given no harvesting and $\omega_k = \omega_m = 0$, (A5) implies

$$\mathrm{X}_1 = \frac{\alpha\gamma^{m\eta+1}\left(1-\beta\right)}{\alpha^{\eta}\gamma^{m\eta+1} - \alpha\beta\gamma^{m+\eta}},$$

i.e. without harvesting the population is viable iff $\alpha^{\eta}\gamma^{m\eta+1} - \alpha\beta\gamma^{m+\eta} > 0$, or equivalently

$$1 < \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}. \qquad \text{(A6)}$$

Under harvesting the existence of strictly positive steady state requires that the denominator of (A5) is strictly positive and $1 - n_k\omega_k > 0, 1 - n_m\omega_m > 0$. The denominator of (A5) is zero iff $\omega_k = 1/n_k$ or

$$\omega_k n_k = 1 - \alpha^{-1}\beta^{1/(\eta-1)}\gamma^{1-m}\left(1 - n_m\omega_m\right)^{-1}. \qquad \text{(A7)}$$

By (A6) and (A7), $\omega_k > 0$ when $\omega_m = 0$, $\omega_k = 0$ when $\omega_m = 1/n_m$ and $\mathrm{X}_1 > 0$ when $\omega_k = \omega_m = 0$ implying that (A7) defines ω_k as a decreasing concave function of ω_m. Since, $x_1 = \mathrm{X}^{1/(1-\eta)}$, we obtain a necessary and sufficient condition for $x_1 > 0$ as

$$\left(1 - \omega_k n_k\right)\left(1 - n_m\omega_m\right) > \alpha^{-1}\beta^{1/(\eta-1)}\gamma^{1-m} \qquad \text{(A8)}$$

After elimination of ω_k and ω_m by (26) and (27) we obtain (28). The population steady state, $x_s, s = 2,...,m$ in (30), follows from (10).□

Proof of Proposition 2.

In matrix notation the linearly transformed system (A1), (A2) is written as $\mathbf{X}_{t+1} = \mathbf{G}\mathbf{X}_t + \mathbf{X}$, where $\mathbf{X}_t = \left[\mathrm{X}_{1t}, \mathrm{X}_{2t}, ..., \mathrm{X}_{mt}\right]$, $\mathbf{X} = \left[\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_m\right]$ and

$$
\mathbf{G}=\begin{bmatrix}
0 & 0 & \dots & 0 & \alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta} \\
\gamma^{1-\eta}\mu_1 & 0 & \dots & 0 & 0 \\
0 & \gamma^{1-\eta}\mu_2 & \cdots & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
\vdots & & & & \\
0 & 0 & \cdots & \gamma^{1-\eta}\mu_{m-1} & 0
\end{bmatrix}, \tag{A9}
$$

given that $\mu_s = 1 - n_k\omega_k$, $s = 1,...m-1$ for $s = k$, $1 < k < m-1$ and $\mu_s = 1$ for $s \neq k$.

Computing the Jacobian matrix for $m = 2,3,...$ shows that the characteristic equation for (A9) can be written as

$$
\tau^m - \alpha^{1-\eta}\beta\left(1-n_m\omega_m\right)^{1-\eta}\left(1-n_k\omega_k\right)^{1-\eta}\gamma^{(1-m)(1-\eta)} = 0. \tag{A10}
$$

Given the existence of positive steady state (condition (45)), the moduli of roots τ in (A10) are strictly below one implying that the steady state is globally asymptotically stable for optimal feedback solutions.☐

Appendix C. Proof of proposition 3.

a) From (26), (27) and (29) we obtain $\partial\omega_k / \partial u_k > 0, \partial\omega_m / \partial u_k < 0$, $\partial\omega_k / \partial u_m < 0, \partial\omega_m / \partial u_m > 0$ and $\partial x_1 / \partial u_k = 0$. The remaining claims follow from (30). b) From (26), (27) and (29) we obtain $\partial\omega_k / \partial n_k < 0, \partial\left(\omega_k n_k\right) / \partial n_k > 0, \partial\omega_m / \partial n_k < 0$, $\partial\omega_m / \partial n_m < 0, \partial\left(\omega_m n_m\right) / \partial n_m > 0, \partial\omega_k / \partial n_m < 0$ and $\partial x_1 / \partial n_k = 0$. The remaining claims then follow from (30). c) From (29) $\partial\lambda / \partial\rho > 0$ and $\partial x_1 / \partial\lambda > 0$ and by (26), (27) $\partial\omega_s / \partial\rho < 0, s = k, s$ implying that $\partial x_s / \partial\rho > 0, s = 1,...,m$. Finally, the effects of ρ on h_k and h_m follow by (28). ☐

Proof of Proposition 5.

Assume $u_k = 0$ and set $\omega_k = 0$ in $y_1(\omega_k, \omega_m) = 0$ and write

$$y_1(0, \omega_m) = \frac{\alpha^{\eta-1}\gamma^{(m-1)(\eta-1)}\rho^{-m}(1 - n\omega_m)^{\eta}}{\beta} + n\omega_m - \omega_m - 1 = 0 . \qquad \text{(A11)}$$

Eliminating ω_m from (A11) by $y_2(0, \omega_m) = 0$ and solving (A11) for ρ produces

$$\rho = \beta^{\frac{1}{m(\eta-1)}}\left[\beta^{\frac{1}{n-1}}(n-1)/n + \alpha\gamma^{m-1}/n\right]^{\frac{-1}{m}} . \qquad \text{(A12)}$$

Since $\left.\frac{\partial \omega_m}{\partial \rho}\right|_{y_2(0,\omega_m)=0} < 0$ and $\frac{\partial y_2}{\partial \omega_m} < 0$, the level of ρ must be higher than the RHS of (A12) which is equivalent to condition (46). The symmetry of (43) and (45) implies that the argument for the case $u_m = 0$ is exactly analogous.□

Proof of Proposition 6.

By Proposition 5 $y_1(\omega_k, \omega_m) = y_2(\omega_k, \omega_m) = 0$ given $n(\rho^{-m} - 1) = \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1} - 1$ and either $\omega_k = 0$ or $\omega_m = 0$. After elimination ω_k from $y_1(\omega_k, \omega_m)$ by $y_2(\omega_k, \omega_m) = 0$, the equation $y_1(\omega_k, \omega_m) = 0$ becomes a function of ω_m, say $y(\omega_m)$. It holds that $y(0) = 0, y'(0) < 0$ and $y''(\omega_m) > 0$ implying by $\partial y_1 / \partial \omega_k > 0$ that the solution pair of $y_1(\omega_k, \omega_m) = 0$ exists above the solution pair of $y_2(\omega_k, \omega_m) = 0$ when $\omega_m > 0, \omega_k > 0.$ Decreasing the level of ρ implies (47) and shifts the solution locus of $y_1(\omega_k, \omega_m) = 0$ upwards implying that no steady states exist.□

Proof of Proposition 7.

Positive steady states exist if the locus of $y_1(\omega_m,\omega_k)=0$ is below the locus of $y_2(\omega_m,\omega_k)=0$. By the symmetry, in the borderline case both equations are satisfied only at one single point where $\omega_m=\omega_k$. Equation $y_2(\omega_m,\omega_k)=0$ yields two solutions for such a point, i.e.

$$\omega_k=\frac{1}{n}\pm\frac{\beta^{\frac{1}{2(\eta-1)}}\gamma^{\frac{1-m}{2}}}{n\sqrt{\alpha}}. \tag{A13}$$

Elimination of ω_m and ω_k from $y_1(\omega_m,\omega_k)=0$ by $\omega_m=\omega_k$ and the lower root in (A13) enables to obtain a nonnegative solution for ρ, which can be given as (48) when taken as an equality. After elimination of ω_k by $y_2(\omega_k,\omega_m)=0$ from $y_1(\omega_m,\omega_k)$ we obtain a function of ω_m and its value is strictly negative excluding the lower root of (A13). Since $\partial y_1/\partial\omega_k>0$, the locus of $y_1(\omega_m,\omega_k)=0$ cannot exceed the locus of $y_2(\omega_m,\omega_k)=0.$ Increasing ρ shifts the locus of $y_2(\omega_m,\omega_k)=0$ downwards. By the fact that the LHS of (A13) is decreasing in ρ implies that (48) is sufficient to $x_s>0, s=1,...,m$ at the Nash equilibrium steady state.□

Proof of Corollary 1.

The Nash equilibrium pair ω_m and ω_k solve (42) and (43). Equation (43) defines ω_k as a decreasing function of ω_m (Proof of Proposition 4) that is independent of u_k, u_m and k. Equation (42) determines ω_k as shown in (44) and we obtain $\partial\omega_k/\partial u_k>0$ and $Sign(\partial\omega_k/\partial k)=Sign(ln(\rho)-(n-1)ln(\gamma))$ $=Sign(\rho\gamma^{1-\eta}-1)$. This directly determines the dependence of ω_k and ω_m on u_k/u_m and k. When $\rho_1<p<\rho_2$, Propositions 5 and 7 imply that the solution lines for $y_1(\omega_m,\omega_k)=0$ and $y_2(\omega_m,\omega_k)=0$

cross. This implies that the existence of the Nash equilibrium with $x_s > 0, s = 1,...,m$ becomes dependent on u_k / u_m and k. □

Proof of Proposition 8

Write conditions (41) and (42) as

$$g_1(\omega_k, \omega_m, \mu) = \frac{u_m \delta_k \omega_m^{-\eta} \alpha^{\eta-1} (1 - n\omega_m)^\eta}{\beta\rho} - u_k \omega_k^{-\eta} (1 + \omega_k - \mu n \omega_k) = 0, \quad \text{(A14)}$$

$$g_2(\omega_k, \omega_m, \mu) = \frac{u_k \delta_m \omega_k^{-\eta} \left[\gamma (1 - n\omega_k)\right]^\eta}{\gamma\rho} - u_m \omega_m^{-\eta} (1 + \omega_m - \mu n \omega_m) = 0, \quad \text{(A15)}$$

where $0 \le \mu \le 1$. When $\mu = 0$, these equations coincide equations (24) and (25), i.e. define the collectively optimal outcome. Assume $\mu = 0$ and solve from (A14) and (A15)

$$\omega_{ko} = \frac{\omega_{mo} \alpha^{(1-\eta)/\eta} \left(\dfrac{\beta \rho u_k}{u_m \delta_k} \right)^{\frac{1}{\eta}}}{1 - n\omega_{mo}}, \quad \text{(A16)}$$

$$\omega_{ko} = \frac{\gamma \omega_{mo} (u_k \delta_m)^{\frac{1}{\eta}}}{(\gamma \rho u_m)^{\frac{1}{\eta}} + n\gamma \omega_{mo} (u_k \delta_m)^{\frac{1}{\eta}}}. \quad \text{(A17)}$$

Equation (A16) defines ω_{ko} as an increasing strictly convex function of ω_{mo} and (A17) an increasing and strictly concave function respectively. Both functions imply that $\omega_{ko} = 0$ when $\omega_{mo} = 0$. At the origin the slope of (A17) exceeds the slope of (A16) implying the existence of a unique ω_{ko}, ω_{mo} solving (A14), (A15). From (A14) $\partial g_1 / \partial \omega_k > 0, \partial g_1 / \partial \mu > 0$ implying $d\omega_k / d\mu < 0$, i.e. in the ω_m, ω_k plane increases of μ shifts the locus of $g_1(\omega_k, \omega_m, \mu) = 0$ downwards. From (A15)

$\partial g_2 / \partial \omega_k < 0, \partial g_2 / \partial \mu > 0$ implying $d\omega_k / d\mu > 0$, i.e. in the ω_m, ω_k plane increases of μ shifts the locus of $g_2(\omega_k, \omega_m, \mu) = 0$ upwards. This implies $\omega_{io} < \omega_{in}, i = k, m.$ □

Proof of Proposition 9.

By (59) $\omega_k = 0$ when $\omega_m = 0$ and ω_k is an increasing function of ω_m. By (57) $q_m(0,0) < 0$ if $\Phi - 1 < 0$. By (58) $\Phi - 1 < 0$ is equivalent to $\rho^{-m} - \beta\alpha^{1-\eta}\gamma^{(1-\eta)(m-1)} > 0$ which holds by $\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1} > 1.$ Additionally, $q_m(\omega_k, 0) = 0$ when $\omega_k = \left(1 - \Phi^{\frac{1}{\eta-1}}\right) n_k^{-1} < n_k^{-1}$. By $\Phi - 1 < 0$, $q_m(0, \omega_m) = 0$ holds with some finite level of $0 < \hat{\omega}_m < n_m^{-1}$ since $q_m(0,0) < 0$ and $q_m(0, \omega_m) = -(1 - n_m\omega_m)^\eta (1 - n_m\omega_m + \omega_m)^{-1} + \Phi$ is an increasing function of ω_m and $q_m(0, n_m^{-1}) > 0$. Because $q_m = 0$ defines ω_k as a monotonically decreasing function of ω_m in the domain $0 \le \omega_m \le \hat{\omega}_m$ the solution exists and is unique.□

Proof of Proposition 10.

When written as an equality (64) follows directly from (61)-(63). The inequality sign follows by (45) and since the LHS of (60) decreases in ω_k and because $\partial q_k / \partial\left(\rho^{-m}\right) < 0$ and $\partial q_k / \partial \omega_k > 0$ in (57). When ρ^{-m} is increased from the level that satisfies (64) as an equality, the level of ω_k or ω_m must increase in order to (63) to be satisfied. This, implies that in (60) the RHS exceeds the LHS implying by (45) that the steady state with $x_s > 0, s = 1, ..., m$ does not exist.

The discount factor comparison holds true if

$$\frac{1}{n}\left(\sqrt{\alpha}\beta^{\frac{1}{2(1-\eta)}}\gamma^{\frac{m-1}{2}}1 - n\right) - \alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1} < 0. \tag{A18}$$

The sign of the LHS of (A18) equals the $sign\left(\sqrt{\gamma}\beta^{\frac{1}{2(\eta-1)}}-\sqrt{\alpha}\gamma^{\frac{m}{2}}\right)$. Condition (28) implies $1<\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}$.

Since the term $\sqrt{\gamma}\beta^{\frac{1}{2(\eta-1)}}-\sqrt{\alpha}\gamma^{\frac{m}{2}}$ is decreasing in α and zero when $1=\alpha\beta^{\frac{1}{1-\eta}}\gamma^{m-1}$, it follows that the sign in (A18) holds and the level of discount factor satisfying (64) as an equality must be higher than the corresponding discount factor in the case of the socially optimal solution.□

Proof of Proposition 13.

After eliminating u_m / u_k equations (24), (25) imply

$$-(1-n_k\omega_k)^{\eta}(1-n_m\omega_m)^{\eta}+\Phi=0, \tag{A19}$$

where Φ is given by (58). The Nash equilibrium must solve (57) and (58) specified for $i=k$ and this condition can be written as

$$-\frac{(1-n_k\omega_k)^{\eta}(1-n_m\omega_m)^{\eta}}{(1-n_m\omega_m)\left[1-\omega_k(n_k-1)\right]}+\Phi=0. \tag{A20}$$

Given $\omega_k>0,\omega_m>0$ it follows that $0<(1-n_m\omega_m)\left[1-\omega_k(n_k-1)\right]<1$. Since $-(1-n_k\omega_k)(1-n_m\omega_m)$ is increasing in ω_k and ω_m, the locus of (A19) must exist below the locus of (A20) when $\omega_k>0,\omega_m>0$. This rules out the case where both ω_m and ω_k are lower in the Nash equilibrium compared to the collectively optimal solution.□

References

Assmuth, A., Ramo, J. and Tahvonen, O. (2017) Economics of size-structured forestry with carbon storage. Canadian Journal of Forest Research 48, 11-22.

Begon, M., Townsend, C.R. and Harper, J.L. 2006. Ecology: from individuals to ecosystems. Blackwell publishing, fourth edition, Malden, MA, USA.

R.J.H. Beverton, S.J. Holt, 1957. On the dynamics of exploited fish populations, Fish Invest. Ser. II, Mar. Fish G.B. Minist. Agric. Fish. Food 19.

Caswell, H. 2001. Matrix population models. Massachusetts: Sinauer

Clark, C.W., 1973. The economics of overexploitation. Science 181, 630-634.

Clark, C.W., 1990. Mathematical bioeconomics: the optimal management of renewable resources. Second edition, John Wiley & Sons, Inc., New York.

Cushing, D.H. 1973. Dependence of recruitment on parent stock. Journal of the Fisheries Research Board of Canada, 30(12), 1965-1976.

Deriso, R.B., 1980. Harvesting strategies and parameter estimation for an age-structured model. Canadian Journal of Fisheries and Aquatic Sciences 37: 268-282.

Diekert, F., Hjermann, D., Nævdal, E., Stenseth, N., 2010a. Non-cooperative exploitation of multi-cohort Fisheries: the role of gear selectivity in the North-East Arctic cod fishery. Resource and Energy Economics 32(1), 78-92.

Diekert, F., Hjermann, D., Nævdal, E., Stenseth, N., 2010b. Spare the young fish: Optimal harvesting policies for North-East Arctic cod. Environmental and Resource Economics 47(4), 455-475.

Dockner, E., Jørgensen, S. Van Long, N. and Sorger, G., 2000. Differential games in economics and management science. Cambridge University Press, Cambridge, UK.

Doyen, L., Cissé, A. A., Sanz, N., Blanchard, F., Pereau, J.-C., Mar, J.-C., 2018. The tragedy of open ecosystems. Dynamic Games and Applications 8 (1), 117-140.

Fischer, R. D., Mirman, L. J., (1992). Strategic dynamic interaction: Fish wars. Journal of Economic Dynamics and Control 16 (2), 267-287.

Fischer, R. D., Mirman, L. J., 1996. The compleat fish wars: Biological and dynamic interactions. Journal of Environmental Economics and Management 30 (1), 34-42.

Getz, W. 1980. The ultimate-sustainable-yield problem in nonlinear age-structured populations. Mathematical Biosciences 48, 279-292.

Gordon, H.S. 1954. The economic theory of a common property resource: the fishery. Journal of Political Economy 80, 124-142.

Hannesson, R. (1975) Fishery dynamics: A North Atlantic cod fishery. Canadian Journal of Economics 8 (2), 151-173.

Jennings, S. Kaiser, M.J. and Reynolds, J.D. 2001. Marine fisheries ecology. Blackwell Science Ltd, Malden, MA, USA.

Hilborn, R. and Walters, C.J. 2001. Quantitative Fisheries Stock Assessment: Choice Dynamics and Uncertainty, Chapman & Hall, London.

Kapaun, U., Quaas, M. F., 2013. Does the optimal size of a fish stock increase with environmental uncertainties? Environmental and Resource Economics 54 (2), 293-310.

Lehtinen, A and Kuorikoski, J,. 2007. Computing the perfect model: why do economists shun simulation. Philosophy of Science 74(3), 304-329.

Levhari, D., Mirman, L. J., 1980. The great fish war: An example using a dynamic Cournot-Nash solution. The Bell Journal of Economics 11(1), 322-334.

Mitra, T., Sorger, G., 2014. Extinction in common property resource models: an analytically tractable example. Economic Theory 57(1), 41-57.

Myers, R.A., Bowen, K.G. and Barrowman, N.J. (1999) Maximum reproductive rate of fish at low population sizes. Canadian Journal of Fish. Aquat. Sci. 56: 2404-2419.

Pekkarinen, A-J., Kumpula, J. and Tahvonen, O. 2015. Reindeer management and winter pastures in the presence of suplementary feeding and government subsidies. Ecological Modelling, 312: 256-271.

Olsen, E.M, Heino, M, Lilly, G.R, Morgan, MJ, Brattey, J, Ernande B. and Dieckmann, U. 2004. Maturation trends indicative of rapid evolution preceded the collapse of northern cod. Nature 428:932–935

Quaas, M. F., Requate, T., Ruckes, K., Skonhoft, A., Vestergaard, N., Voss, R., 2013. Incentives for optimal management of age-structured fish populations. Resource and Energy Economics 35 (2), 113-134.

Quaas, M. F. and Tahvonen, O. (2018) Strategic harvesting of age-structured populations, manuscript, Department of Economics, Leipzig University, Germany.

Reed, W.J. 1980. Optimum age-specific harvesting in a nonlinear population model. Biometrics 36, 579-593.

Roff, D.A. (1992) The evolution of life histories: theory and analysis. Chapman and Hall, New York.

Samuelson, P. (1977). Generalizing Fisher's "reproduction value": linear differential and difference equations of "dilute" biological systems. Proceedings of the National Academy of Sciences of the United States of America, 74, 5189–5192.

Tahvonen, O., 2009. Economics of harvesting age-structured fish populations. Journal of Environmental Economics and Management 58(3), 281-299.

Sorger, G. 2015. Dynamic economic analysis: deterministic models in discrete time. Cambridge University Press, Cambridge, UK.

Tahvonen, O., 2014. Optimal harvesting of size-structured biological populations. In: E. Moser et al. (Ed.), Dynamic Optimization in Environmental Economics. Springer, Berlin Heidelberg.

Tahvonen, O., 2015. Economics of naturally regenerating, heterogeneous forests. Journal of the Association of Environmental and Resource Economists 2(2), 309-337.

Wilen, J. E. (1985). Bioeconomics of renewable resource use. In A. V. Kneese & J. L. Sweeney (Eds.), Handbook of natural resource and energy economics (Vol. I, pp. 61–124), Amsterdam: Elsevier.